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On the hyperbolicity of a model for polyatomic gases in a gravitational field

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Abstract

Einstein Equations aren't hyperbolic because they are invariant under an invertible change of 4-dimensional variables. A possible solution of this problem is to consider a particular set of this 4-dimensional variables in order to reduce the number of the unknowns appearing in the metric tensor. The choice of these variables depends on the particular physical situation where we are working. In fact, in the right hand side of Einstein Equations there is the energy-momentum tensor of the sources; if this is all the matter contained in the Universe, then the problem becomes too complicated to deal with. An approximation can be used in particular situations. For example here the situation is considered of a polyatomic gas generating its own gravity field and sufficiently far from the other matter, so as not to be affected by its influence on the metric tensor. The isotropy of the Universe is imposed by using the Representation Theorems jointly with another change of 4-dimensional variables so as to reduce the unknowns appearing in the 10 components of the metric tensor to only 2 scalar functions. In this way hyperbolic is achieved.

1 Introduction

In the article [1], the hyperbolicity of Einstein Equations have been studied by using armonic coordinates and limiting to the case of Euler Equations for the matter. Here we study this problem by using the isotropy and homogeneity of the universe and in the case of a model for polyatomic gases with many moments. I think that the present work also generalizes the recently published one [2].
Obviously, Einstein Equations aren't hyperbolic because they are invariant under an invertible change of 4-dimensional variables. The solution of this problem is to consider a particular set of this 4-dimensional variables in order to reduce the number of the unknowns appearing in the metric tensor. The choice of these variables depends on the particular physical situation where we are working. In fact, Einstein Equations are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\,\pi\, G}{c^4}\, T_{\mu\nu}\,, \tag{1}$$

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where

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu} \Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} , \quad \text{(Ricci tensor)} ,$$
$$R = g^{\mu\nu} R_{\mu\nu} , \quad \text{(Curvature scalar)} ,$$
$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\tau} \left(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu} \right) , \quad \text{(Christoffel's symbols)} .$$

Moreover, G is the cosmological constant and $T_{\mu\nu}$ the energy-momentum tensor of the sources of the Gravity Field. So (1) is a system of 10 equations in the 10 unknows $g_{\mu\nu}$. If we consider as sources all the matter contained in the Universe, then the problem becomes too complicated to deal with (In particular for the expression of $T_{\mu\nu}$). An approximation can be used in particular situations. For example, near a black hole we can assume that the contribution of the black hole is predominant over all the others and, then, the latter can be neglected; so in this case we can consider the Schwarzschild metric as solution of Einstein Equations and consider only the field equations of the matter but under the influence of the external gravitational field.

Another phisical situation is that of a polyatomic gas generating its own gravity field and sufficiently far from the other matter, so as not to be affected by its influence on the metric tensor. In this case $T_{\mu\nu}$ is the energy-momentum tensor of the polyatomic gas.

We may consider also the case where the sources are all the matter contained in the Universe but only if we assume that all the Universe behavies as a polyatomic gas (or a monoatomic gas as a limiting case). In this case the results are valid only within the limits imposed by this strong approximation.

In any case, the isotropy of the Universe can be easily imposed by using the Representation Theorems; in fact, in this case and in the reference frame comoving with the fluid, the unknown metric tensor $g_{\alpha\beta}$ depends only on the scalar $x^0 = c\,t$ (t is time and c the light speed in vacuum) and on the vector x^i. By applying the Representation Theorems we see that

- Since g_{00} is a scalar, it can be expressed as a function of x^0 and of $s = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, i.e., $g_{00} = g_{00}\left(x^0 , s\right)$.
- Since g_{0i} is a vector, it can be expressed as $g_{0i} = g_1\left(x^0 , s\right) \frac{x_i}{s}$ where g_1 is a scalar function.
- Since g_{ij} is a symmetric tensor, it can be expressed as $g_{ij} = g_2\left(x^0 , s\right) \frac{x_i\, x_j}{s^2} + g_3\left(x^0 , s\right) \delta_{ij}$, where g_2 and g_3 are scalar functions.

In this way we have only the 4 unknows g_{00}, g_1, g_2, g_3 instead of all the 10 independent components of $g_{\mu\nu}$ and the metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} g_{00}\left(x^0 , s\right) & g_1\left(x^0 , s\right) \frac{x_j}{s} \\ g_1\left(x^0 , s\right) \frac{x_i}{s} & g_2\left(x^0 , s\right) \frac{x_i\, x_j}{s^2} + g_3\left(x^0 , s\right) \delta_{ij} \end{pmatrix} . \tag{2}$$

Moreover, the line element is

$$\delta\, s^2 = g_{00} \left(d\, x^0\right)^2 + 2\, g_1\, d\, x^0\, d\, s + g_2\, (d\, s)^2 + g_3\, \delta_{ij}\, d\, x^i\, d\, x^j . \tag{3}$$

This is an isotropic and also a rotational invariant so respecting both the isotropy and the homogeneity of the Universe.

Since Einstein's Equations are invariant under an invertible change of 4-dimensional coordinates, we can use this property to furher reduce the number of the unknowns; for example, we can change variables with the law

$$\begin{cases} x^0 = f\left(X^0 \, , \, S\right) \, , \\ \\ x^i = g\left(X^0 \, , \, S\right) \, \frac{X^i}{S} \end{cases} \, , \tag{4}$$

where $S = \sqrt{X^i X^j \delta_{ij}}$ from which it follows $s = g$. In the next section we will prove that a change of 4-dimensional coordinates of the type (4) can be found such that, in the new coordinates the metric tensor takes the form

$$G_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ \\ 0 & G_2\left(X^0 \, , \, S\right) \, \frac{X_i X_j}{S^2} \, + \, G_3\left(X^0 \, , \, S\right) \, \delta_{ij} \end{pmatrix} \, , \tag{5}$$

which has the form (2) but with $g_{00} = 1$, $g_1 = 0$. In this way the unknowns functions reduce to two. As an exercise, the case of the Schartzschild metric is considered.
In sect. 3 we will calculate the left hand side of Einstein Equations (1), while in sect. 4 we will exploit their right hand side. In sect. 5 we will study the hyperbolicity of eqs. (1) and those for the polyatomic gas; we will see that some equations are consequences of the others and of suitable boundary conditions. By eliminating these differential constraints, the remaining equations give an hyperbolic set of partial differential equations. Moreover, in sect. 6 we will find the boundary values under which Einstein Equations give as result the Friedmann-Robertson-Walker (FRW) metric for flat, open or closed spacetime.

2 Reduction of the unknowns scalar functions in the metric tensor

The line element with the change of 4-dimensional variables $x^\alpha = x^\alpha\left(X^\mu\right)$ becomes

$$\delta \, s^2 = g_{\alpha\beta} \, d \, x^\alpha \, d \, x^\beta = G_{\mu\nu} \, d \, X^\mu \, d \, X^\nu \quad \text{with} \quad G_{\mu\nu} = g_{\alpha\beta} \, \frac{\partial \, x^\alpha}{\partial \, X^\mu} \, \frac{\partial \, x^\beta}{\partial \, X^\nu} \, , \tag{6}$$

and $G_{\mu\nu}$ is the metric tensor in the new variables. In particular, we have

$$\begin{aligned} G_{00} &= g_{00} \left(\frac{\partial \, x^0}{\partial \, X^0} \right)^2 + 2 \, g_{0h} \, \frac{\partial \, x^0}{\partial \, X^0} \, \frac{\partial \, x^h}{\partial \, X^0} + g_{hk} \, \frac{\partial \, x^h}{\partial \, X^0} \, \frac{\partial \, x^k}{\partial \, X^0} \, , \\ G_{0b} &= g_{00} \, \frac{\partial \, x^0}{\partial \, X^0} \, \frac{\partial \, x^0}{\partial \, X^b} + g_{0h} \left(\frac{\partial \, x^0}{\partial \, X^0} \, \frac{\partial \, x^h}{\partial \, X^b} + \frac{\partial \, x^0}{\partial \, X^b} \, \frac{\partial \, x^h}{\partial \, X^0} \right) + g_{hk} \, \frac{\partial \, x^h}{\partial \, X^0} \, \frac{\partial \, x^k}{\partial \, X^b} \, , \\ G_{ab} &= g_{00} \, \frac{\partial \, x^0}{\partial \, X^a} \, \frac{\partial \, x^0}{\partial \, X^b} + 2 \, g_{0h} \, \frac{\partial \, x^0}{\partial \, X^{(a}} \, \frac{\partial \, x^h}{\partial \, X^{b)}} + g_{hk} \, \frac{\partial \, x^h}{\partial \, X^a} \, \frac{\partial \, x^k}{\partial \, X^b} \, . \end{aligned} \tag{7}$$

With the particular metric (2) and the change of variables (4), these expressions become

$$G_{00} = g_{00} \left(\frac{\partial f}{\partial X^0} \right)^2 + 2\, g_1 \frac{\partial f}{\partial X^0} \frac{\partial g}{\partial X^0} + (g_2 + g_3) \left(\frac{\partial g}{\partial X^0} \right)^2 , \; G_{0b} = G_1 \frac{X_b}{S} ,$$
$$\text{with} \quad G_1 = g_{00} \frac{\partial f}{\partial X^0} \frac{\partial f}{\partial S} + g_1 \left(\frac{\partial f}{\partial S} \frac{\partial g}{\partial X^0} + \frac{\partial f}{\partial X^0} \frac{\partial g}{\partial S} \right) + (g_2 + g_3) \frac{\partial g}{\partial S} \frac{\partial g}{\partial X^0} ,$$
$$G_{ab} = G_2 \frac{X_a X_b}{S^2} + G_3\, \delta_{ab} \quad \text{with}$$
$$G_2 = g_{00} \left(\frac{\partial f}{\partial S} \right)^2 + 2\, g_1 \frac{\partial f}{\partial S} \frac{\partial g}{\partial S} + (g_2 + g_3) \left(\frac{\partial g}{\partial S} \right)^2 - g_3 \frac{g^2}{S^2} , \quad G_3 = g_3 \frac{g^2}{S^2} . \tag{8}$$

We aim to prove that a change of variables $x^\alpha = x^\alpha (X^\mu)$ exists such that $G_{00} = 1$, $G_1 = 0$. The proof become easier if we firstly prove the following

LEMMA: "For any given value of $g_{\alpha\beta}$ expressed by (2), the functions $F \left(x^0 , s\right)$, $G \left(x^0 , s\right)$, $G_2 \left(x^0 , s\right)$, $G_3 \left(x^0 , s\right)$ exist such that

$$\begin{aligned} g_{00} &= \left(\frac{\partial F}{\partial x^0} \right)^2 + (G_2 + G_3) \left(\frac{\partial G}{\partial x^0} \right)^2 , \\ g_1 &= \frac{\partial F}{\partial x^0} \frac{\partial F}{\partial s} + (G_2 + G_3) \frac{\partial G}{\partial s} \frac{\partial G}{\partial x^0} , \\ g_2 &= \left(\frac{\partial F}{\partial s} \right)^2 + (G_2 + G_3) \left(\frac{\partial G}{\partial s} \right)^2 - G_3 \frac{G^2}{s^2} , \quad g_3 = G_3 \frac{G^2}{s^2} \text{''}. \end{aligned} \tag{9}$$

Proof: Let us consider the first order quasi-linear partial differential equation in the unknown $\eta \left(x^0 , s\right)$:

$$\begin{aligned} &\left(\eta \frac{\partial \eta}{\partial s} - \frac{\partial \eta}{\partial x^0} \right) \left[(g_1)^2 - g_{00} (g_2 + g_3) \right] + \eta^2 \frac{\partial g_1}{\partial s} \left[g_1 - \eta (g_2 + g_3) \right] + \eta\, g_1 \frac{\partial g_1}{\partial x^0} + \\ &+ \eta \frac{\partial g_{00}}{\partial s} \left[-\frac{3}{2} g_1 + \eta (g_2 + g_3) \right] - \frac{1}{2} \eta \frac{\partial g_{00}}{\partial x^0} (g_2 + g_3) - \frac{1}{2} (g_{00} - \eta\, g_1) \eta^2 \frac{\partial (g_2 + g_3)}{\partial s} + \\ &+ \eta \left[g_{00} - \frac{3}{2} \eta\, g_1 + \frac{1}{2} \eta^2 (g_2 + g_3) \right] \frac{\partial (g_2 + g_3)}{\partial x^0} = \left(\sqrt{g_{00}} \right)^3 \left(\frac{\partial}{\partial x^0} \frac{g_1}{\sqrt{g_{00}}} - \frac{\partial}{\partial s} \sqrt{g_{00}} \right) . \end{aligned} \tag{10}$$

We note that the coefficient $\left[(g_1)^2 - g_{00} (g_2 + g_3) \right]$ isn't zero because from (2) we have $det\, g_{\alpha\beta} = (g_3)^2 \left[g_{00} (g_2 + g_3) - (g_1)^2 \right] < 0$. Moreover, we choose for this equation a boarding condition satisfying the relation

$$\frac{g_1 + \sqrt{(g_1)^2 - g_{00} (g_2 + g_3)}}{g_2 + g_3} < \eta < \frac{g_1 - \sqrt{(g_1)^2 - g_{00} (g_2 + g_3)}}{g_2 + g_3} , \tag{11}$$

so that, for continuity reason, it will be satisfied also in a neighbourhood of the initial variety. After that, we find the function F by integrating the equations

$$\begin{cases} \frac{\partial F}{\partial x^0} = \frac{|g_{00}-\eta\, g_1|}{\sqrt{\eta^2(g_2+g_3)-2\,\eta\, g_1+g_{00}}}\,, \\ \frac{\partial F}{\partial s} = \frac{|g_{00}-\eta\, g_1|}{g_{00}-\eta\, g_1}\,\frac{g_1-\eta\,(g_2+g_3)}{\sqrt{\eta^2(g_2+g_3)-2\,\eta\, g_1+g_{00}}}\,, \end{cases} \tag{12}$$

We note that $\eta^2\,(g_2+g_3) - 2\,\eta\, g_1 + g_{00} > 0$ as consequence of (11); moreover, by calculating this expression in $\eta = \frac{g_{00}}{g_1}$ it becomes

$$\left(\frac{g_{00}}{g_1}\right)^2 (g_2+g_3) - g_{00} < 0\,,$$

so that $g_{00} - \eta\, g_1 \neq 0$ in the interval (11). Finally, the integrability condition on (12) is nothing more than (10). Consequently, eq. (12) has certainly a solution F.
Now we can find the function G by solving the first order quasi-linear partial differential equation

$$\frac{\partial G}{\partial x^0} = \eta\,\frac{\partial G}{\partial s}\,, \tag{13}$$

and consider for G_2 and G_3 the expressions

$$G_2+G_3 = \frac{g_2+g_3-\left(\frac{\partial F}{\partial s}\right)^2}{\left(\frac{\partial G}{\partial s}\right)^2}\,, \quad G_3 = g_3\left(\frac{s}{G}\right)^2\,. \tag{14}$$

Now that we have all the ingredient, we can prove ours eqs. (9); let us begin with $(9)_1$:
By using $(12)_1$, (14) and (13) we see that

$$\left(\frac{\partial F}{\partial x^0}\right)^2 + (G_2+G_3)\left(\frac{\partial G}{\partial x^0}\right)^2 = \frac{(g_{00}-\eta\, g_1)^2}{\eta^2\,(g_2+g_3) - 2\,\eta\, g_1 + g_{00}} + \eta^2\left[g_2+g_3-\left(\frac{\partial F}{\partial s}\right)^2\right] \stackrel{*}{=} g_{00}\,,$$

where in the passage denoted by $\stackrel{*}{=}$ $(12)_2$ has been used. The result proves $(9)_1$.
Let us consider now $(9)_2$: By using (12), (14) and (13) we see that

$$\frac{\partial F}{\partial x^0}\frac{\partial F}{\partial s} + (G_2+G_3)\,\frac{\partial G}{\partial s}\frac{\partial G}{\partial x^0} = \frac{(g_{00}-\eta\, g_1)\,[g_1-\eta\,(g_2+g_3)]}{\eta^2\,(g_2+g_3) - 2\,\eta\, g_1 + g_{00}} + \eta\left[g_2+g_3-\left(\frac{\partial F}{\partial s}\right)^2\right] \stackrel{*}{=} g_1\,,$$

where in the passage denoted by $\stackrel{*}{=}$ $(12)_2$ has been used. The result proves $(9)_2$.
Eq. $(9)_3$ can be easily proven. in fact, by using (14) we see that

$$\left(\frac{\partial F}{\partial s}\right)^2 + (G_2+G_3)\left(\frac{\partial G}{\partial s}\right)^2 - G_3\,\frac{G^2}{s^2} = g_2\,.$$

Finally, $(9)_4$ is a direct consequence of $(14)_2$. We prove now the

THEOREM 1: "A change of 4-dimensional variables $x^\alpha = x^\alpha\,(X^\mu)$ exist such that $G_{00}=1$, $G_1=0$."

To prove this theorem, let $x^0 = f\left(X^0\,,\,S\right)$, $s = g\left(X^0\,,\,S\right)$ be the inverse functions of
$X^0 = F\left(x^0\,,\,s\right)$, $S = G\left(x^0\,,\,s\right)$ with F and G given in the Lemma. By derivation of the composite functions, we obtain

$$\begin{cases} 1 = \frac{\partial f}{\partial X^0}\frac{\partial F}{\partial x^0} + \frac{\partial f}{\partial S}\frac{\partial G}{\partial x^0} \\ 0 = \frac{\partial f}{\partial X^0}\frac{\partial F}{\partial s} + \frac{\partial f}{\partial S}\frac{\partial G}{\partial s} \end{cases}, \qquad \begin{cases} 1 = \frac{\partial g}{\partial X^0}\frac{\partial F}{\partial s} + \frac{\partial g}{\partial S}\frac{\partial G}{\partial s} \\ 0 = \frac{\partial g}{\partial X^0}\frac{\partial F}{\partial x^0} + \frac{\partial g}{\partial S}\frac{\partial G}{\partial x^0} \end{cases}.$$

From these relations we obtain

$$\frac{\partial f}{\partial X^0} = \begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-1} \frac{\partial G}{\partial s}, \quad \frac{\partial f}{\partial S} = -\begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-1} \frac{\partial F}{\partial s},$$
$$\frac{\partial g}{\partial X^0} = -\begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-1} \frac{\partial G}{\partial x^0}, \quad \frac{\partial g}{\partial S} = \begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-1} \frac{\partial F}{\partial x^0}.$$

By using these expressions, $(8)_1$ becomes

$$G_{00} = \begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-2} \left[g_{00}\left(\frac{\partial G}{\partial s}\right)^2 - 2\,g_1 \frac{\partial G}{\partial s}\frac{\partial G}{\partial x^0} + (g_2+g_3)\left(\frac{\partial G}{\partial x^0}\right)^2 \right] = 1\,,$$

where in the last passage eqs. (9) has been used. Similarly, $(8)_3$ becomes

$$G_1 = \begin{vmatrix} \frac{\partial F}{\partial x^0} & \frac{\partial G}{\partial x^0} \\ \frac{\partial F}{\partial s} & \frac{\partial G}{\partial s} \end{vmatrix}^{-2} \left[-\,g_{00}\frac{\partial G}{\partial s}\frac{\partial F}{\partial s} + g_1 \frac{\partial F}{\partial s}\frac{\partial G}{\partial x^0} + g_1 \frac{\partial F}{\partial x^0}\frac{\partial G}{\partial s} - \right.$$
$$\left. (g_2+g_3)\frac{\partial F}{\partial x^0}\frac{\partial G}{\partial x^0} \right] = 0\,,$$

where in the last passage eqs. (9) has been used. This completes the proof of the Theorem.
As example of application of this method, let us consider the Schwarzschild metric outside the mass M generating it and with spherical simmetry and without rotations and charges; it can be found in eq. (12.62) on page 437 of [3], or in eq. (A.1) of [4]. It reads

$$g_{\alpha\beta} = \mathrm{diag}\left[F\left(x^1\right),\, -\frac{1}{F\left(x^1\right)},\, -\left(x^1\right)^2,\, -\left(x^1\right)^2 \sin^2 x^2 \right] \quad \text{with} \quad F = 1 - \frac{2\,G\,M}{c^2 x^1}\,,$$

where G is the gravitational constant. This is a particular case of the present eq. (3) with $g_1 = 0$, and g_{00}, g_2 and g_3 not depending of time. By applying the present approach we have then only to find a transformation of 4-dimensional coordinates which transforms g_{00} to 1. We will see this now. To this end, let us consider the solution η of the first one of the following equations and, after that, a solution G_1 of the second one:

$$\frac{\partial}{\partial x^0}\frac{\eta}{F} = \frac{\partial}{\partial x^1}\ln\left|\sqrt{F}\cosh\eta\right|, \quad \frac{\partial}{\partial x^0}\frac{\cosh\eta}{\sqrt{-F\,G_1}} = \frac{\partial}{\partial x^1}\left(\sqrt{\frac{F}{-G_1}}\sinh\eta\right).$$

By using these functions η and G_1, we see that the following equations are integrable and give the functions f and g:

$$\frac{\partial f}{\partial x^0} = \sqrt{F} \cosh \eta \, , \; \frac{\partial f}{\partial x^1} = \frac{1}{\sqrt{F}} \sinh \eta \, , \; \frac{\partial g}{\partial x^0} = \sqrt{\frac{F}{-G_1}} \sinh \eta \, , \; \frac{\partial g}{\partial x^1} = \frac{\cosh \eta}{\sqrt{-F G_1}} \, .$$

From the above equations it follows that the following system is satisfied:

$$F = \left(\frac{\partial f}{\partial x^0} \right)^2 + G_1 \left(\frac{\partial g}{\partial x^0} \right)^2 , \, 0 = \frac{\partial f}{\partial x^0} \frac{\partial f}{\partial x^1} + G_1 \frac{\partial g}{\partial x^0} \frac{\partial g}{\partial x^1} \, ,$$

$$- \frac{1}{F} = \left(\frac{\partial f}{\partial x^1} \right)^2 + G_1 \left(\frac{\partial g}{\partial x^1} \right)^2 .$$

We also define $G_2 = - \left(x^1\right)^2$, $G_3 = - \left(x^1\right)^2 \sin^2 X^2$, where x^1 is the expression coming from the inverse of $X^0 = f\left(x^0 , x^1\right)$, $X^1 = g\left(x^0 , x^1\right)$. With the change of 4-dimensional variables $X^0 = f\left(x^0 , x^1\right)$, $X^1 = g\left(x^0 , x^1\right)$, $X^2 = x^2$, $X^3 = x^3$ for the line element $d s^2$ we then have

$$\begin{aligned} d s^2 = d \left(X^0\right)^2 + G_1 d \left(X^1\right)^2 + G_2 d \left(X^2\right)^2 + G_3 d \left(X^3\right)^2 = \\ = \left(\frac{\partial f}{\partial x^0} d x^0 + \frac{\partial f}{\partial x^1} d x^1 \right)^2 + G_1 d \left(\frac{\partial g}{\partial x^0} d x^0 + \frac{\partial g}{\partial x^1} d x^1 \right)^2 - \left(x^1\right)^2 d \left(x^2\right)^2 - \\ \left(x^1\right)^2 \sin^2 x^2 d \left(x^3\right)^2 = F d \left(x^0\right)^2 - \frac{1}{F} d \left(x^1\right)^2 - \left(x^1\right)^2 d \left(x^2\right)^2 - \\ \left(x^1\right)^2 \sin^2 x^2 d \left(x^3\right)^2 . \end{aligned}$$

The expression at the end of the above expression is the line element for the above Schwarzschild metric, So we have proved that with a transformation of 4-dimensional variables it takes the form $g_{\alpha\beta} = \text{diag}\,(1 \, , G_1 \, , G_2 \, , G_3)$.

Obviously, this was only an exercise because the original metric has already the diagonal form and, moreover, doesn't depend on x^0, while in the new metric the advantage to have 1 instead of g_{00} is canceled by the fact that it depends on X^0. Moreover, this case goes outside the scopes of the present article because it concerns a metric outside the mass M generating it, so that the right hand side of Einstein Equations (1) is zero and there is no coupling between the metric and the eventual polyatomic gas that generates it. One could study the influence of this metric on a polyatomic gas gravitating around a black hole; in this case the metric is an external field for the gas. In [4] Kremer studied this case for a monoatomic gas; the generalization to a polyatomic gas may be the object of a future article.

Coming back to the general treatment before the Schwarzschild example, if we put ourselves from the beginning in the new coordinates, then (8) can be written as

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & g_2\left(x^0 , s\right) \frac{x_i x_j}{s^2} + g_3\left(x^0 , s\right) \delta_{ij} \end{pmatrix} . \tag{15}$$

So the unknown functions reduce from the four g_{00}, g_1, g_2, g_3 to only two functions, i.e., g_2 and g_3.

A further simplification is obtained by using spherical coordinates

$x_1 = s \cos \vartheta\,, \quad x_2 = s \sin \vartheta \cos \varphi\,, \quad x_3 = s \sin \vartheta \sin \varphi.$
In this case the line element becomes

$$\delta s^2 = \left(d x^0\right)^2 + G_2\, (d s)^2 + G_3 \left[(d \vartheta)^2 + \sin^2 \vartheta\, (d \varphi)^2\right]\,,$$

with $G_2 = g_2 + g_3$, $G_3 = g_3\, s^2$. In this case the metric tensor takes the diagonal form

$$g_{\alpha\beta} = diag\, \left(1\,, G_2\,, G_3\,, G_3 \sin^2 \vartheta\right)\,, \tag{16}$$

with G_2 and G_3 depending on x^0 and s.

3 Calculation of the left hand side of eq. (1) with the metric (16) (coming from (5))

Let us begin with the Christoffel's symbols; we can calculate them directly from their definition $(1)_4$ or with the shorter way indicated in [3], chapter 12.2, page 431. This method can be summarized as follows:
Let us forget the framework in which we are working and consider a problem in the context of Rational Mechanics: We start with x^α as lagrangian parameters and with the Lagrangian

$$\mathcal{L} = g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta\ .$$

In this case the Lagrange equations become

$$2\, g_{\alpha\beta}\, \ddot{x}^{\,\beta} + 2\, (\partial_\gamma\, g_{\alpha\beta})\, \dot{x}^\gamma \dot{x}^\beta - (\partial_\alpha\, g_{\mu\nu})\, \dot{x}^\mu \dot{x}^\nu = 0\,, \quad \text{which, contracted by } g^{\delta\alpha} \text{ gives}$$

$$\ddot{x}^{\,\delta} + M^\delta_{\beta\gamma}\, \dot{x}^\beta \dot{x}^\gamma = 0\,, \quad \text{with} \quad M^\delta_{\beta\gamma} = \frac{1}{2}\, g^{\delta\alpha}\, (2\, \partial_\gamma\, g_{\alpha\beta} - \partial_\alpha\, g_{\gamma\beta})\ .$$

From this result we see that

$$\Gamma^\gamma_{\mu\nu} = M^\gamma_{(\mu\nu)}\,.$$

Another interesting property which facilitates the calculations for the second term in $(1)_2$ is the following one:
Property: "We have that

$$\Gamma^\alpha_{\alpha\mu} = \frac{1}{2}\, \partial_\mu \ln |\det\ g_{\alpha\beta}| \quad ". \tag{17}$$

Let us apply this method to the case where $g_{\alpha\beta}$ is given by $g_{\alpha\beta} = \text{diag}\,(1\,, g_1\,, g_2\,, g_3)$; the expression (16) will be a consequence in the particular case $g_1 = G_2$, $g_2 = G_3$, $g_3 = G_3 \sin^2 \vartheta$. We find

$$\Gamma^0_{\mu\nu} = \text{diag}\left(0\,, -\frac{\partial_0\, g_1}{2}\,, -\frac{\partial_0\, g_2}{2}\,, -\frac{\partial_0\, g_3}{2}\right)\,,$$

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}\,\partial_0\,\ln|g_1| & 0 & 0 \\ \frac{1}{2}\,\partial_0\,\ln|g_1| & \frac{1}{2}\,\partial_1\,\ln|g_1| & \frac{1}{2}\,\partial_2\,\ln|g_1| & \frac{1}{2}\,\partial_3\,\ln|g_1| \\ 0 & \frac{1}{2}\,\partial_2\,\ln|g_1| & -\frac{\partial_1\,g_2}{2\,g_1} & 0 \\ 0 & \frac{1}{2}\,\partial_3\,\ln|g_1| & 0 & -\frac{\partial_1\,g_3}{2\,g_1} \end{pmatrix}.$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\,\partial_0\,\ln|g_2| & 0 \\ 0 & -\frac{\partial_2\,g_1}{2\,g_2} & \frac{1}{2}\,\partial_1\,\ln|g_2| & 0 \\ \frac{1}{2}\,\partial_0\,\ln|g_2| & \frac{1}{2}\,\partial_1\,\ln|g_2| & \frac{1}{2}\,\partial_2\,\ln|g_2| & \frac{1}{2}\,\partial_3\,\ln|g_2| \\ 0 & 0 & \frac{1}{2}\,\partial_3\,\ln|g_2| & -\frac{\partial_2\,g_3}{2\,g_2} \end{pmatrix}.$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\,\partial_0\,\ln|g_3| \\ 0 & -\frac{\partial_3\,g_1}{2\,g_3} & 0 & \frac{1}{2}\,\partial_1\,\ln|g_3| \\ 0 & 0 & -\frac{\partial_3\,g_2}{2\,g_3} & \frac{1}{2}\,\partial_2\,\ln|g_3| \\ \frac{1}{2}\,\partial_0\,\ln|g_3| & \frac{1}{2}\,\partial_1\,\ln|g_3| & \frac{1}{2}\,\partial_2\,\ln|g_3| & \frac{1}{2}\,\partial_3\,\ln|g_3| \end{pmatrix}.$$

where ∂_α denotes $\frac{\partial}{\partial\,x^\alpha}$.
Now we can calculate the Ricci Tensor $(1)_2$ and find

$$R_{00} = -\,\frac{1}{2}\,\partial_{00}\,\ln\,|g_1 g_2 g_3|\,-\,\frac{1}{4}\sum_{A=1}^{3}\left(\partial_0\,\ln\,|g_A|\right)^2\ . \tag{18}$$

$$\begin{aligned} R_{01} = &-\,\frac{1}{2}\,\partial_{01}\,\ln\,|g_2 g_3|\,+\,\frac{1}{4}\,\left(\partial_1\,\ln\,|g_2 g_3|\right)\left(\partial_0\,\ln\,|g_1|\right) \\ &-\,\frac{1}{4}\,\left(\partial_0\,\ln\,|g_2|\right)\left(\partial_1\,\ln\,|g_2|\right)\,-\,\frac{1}{4}\,\left(\partial_0\,\ln\,|g_3|\right)\left(\partial_1\,\ln\,|g_3|\right)\,, \end{aligned}$$

$$\begin{aligned} R_{11} = &-\,\frac{\partial_{00}\,g_1}{2}\,-\,\partial_2\left(\frac{\partial_2\,g_1}{2\,g_2}\right)\,-\,\partial_3\left(\frac{\partial_3\,g_1}{2\,g_3}\right)\,-\,\frac{1}{2}\,\partial_{11}\,\ln\,|g_2 g_3|\,+ \\ &+\,\frac{1}{4}\,\left(\partial_1\,\ln\,|g_2 g_3|\right)\left(\partial_1\,\ln\,|g_1|\right)\,-\,\frac{1}{4}\left(\partial_1\,\ln\,|g_2|\right)^2\,-\,\frac{1}{4}\left(\partial_1\,\ln\,|g_3|\right)^2\,+ \\ &+\,\frac{g_1}{4}\,\left(\partial_0\,\ln\,|g_1|\right)^2 + \frac{g_1}{4\,g_2}\,\left(\partial_2\,\ln\,|g_1|\right)^2 + \frac{g_1}{4\,g_3}\,\left(\partial_3\,\ln\,|g_1|\right)^2 - \,\frac{\partial_0\,g_1}{4}\,\partial_0\,\ln\,|g_2 g_3| - \\ &\frac{\partial_2\,g_1}{4\,g_2}\,\partial_2\,\ln\,|g_2 g_3|\,-\,\frac{\partial_3\,g_1}{4\,g_3}\,\partial_3\,\ln\,|g_2 g_3|\,. \end{aligned}$$

$$R_{12} = -\frac{1}{2}\,\partial_{12}\,\ln|g_3| + \frac{1}{4}\,(\partial_2\,\ln|g_3|)\,(\partial_1\,\ln|g_2|) + \frac{1}{4}\,(\partial_1\,\ln|g_3|)\,(\partial_2\,\ln|g_1|) - \frac{1}{4}\,(\partial_2\,\ln|g_3|)\,(\partial_1\,\ln|g_3|)\,,$$

Moreover, R_{02} can be desumed by $(18)_2$ by exchanging the indexes 1 and 2; similarly, R_{03} can be desumed by $(18)_2$ by exchanging the indexes 1 and 3. In the same way R_{22} and R_{33} can be desumed by $(18)_3$ with a suitable change of indexes, while R_{13} and R_{23} can be desumed by $(18)_4$ with suitable changes of indexes.
Now we can calculate the curvature $(1)_3$; taking into account that

$$\text{the inverse matrix of} \quad g_{\alpha\beta} = \text{diag}\,(1\,,\,g_1\,,\,g_2\,,\,g_3) \quad \text{is} \quad g^{\alpha\beta} = diag\left(1\,,\,\frac{1}{g_1}\,,\,\frac{1}{g_2}\,,\,\frac{1}{g_3}\right)\,, \tag{19}$$

we obtain

$$R = R_{00} + \frac{R_{11}}{g_1} + \frac{R_{22}}{g_2} + \frac{R_{33}}{g_3}\,.$$

So we have now all we need to write the left hand side of Einstein Equation $(1)_1$. But we don't need to write all the components of this equation because they must be coupled with those of the polyatomic gas which, for an N moments model, are

$$\nabla_\alpha V^\alpha = 0\,, \quad \nabla_\alpha T^{\alpha\beta} = 0\,, \quad \nabla_\alpha A^{\alpha\beta_1\cdots\beta_n} = I^{\beta_1\cdots\beta_n}\,, \quad \text{for} \quad n = 2\,,\cdots,N\,. \tag{20}$$

The first two of these equations are the conservation law of mass and that of energy-momentum, respectively; they are contained in $(20)_3$ for $n = 0, 1$ but we have preferred to write them separately for their importance. Moreover, ∇_α denotes the covariant derivative which for a generic tensor $T^{\beta_1\cdots\beta_n}_{\gamma_1\cdots\gamma_n}$ is

$$\nabla_\alpha T^{\beta_1\cdots\beta_n}_{\gamma_1\cdots\gamma_n} = \partial_\alpha T^{\beta_1\cdots\beta_n}_{\gamma_1\cdots\gamma_m} + \sum_{r=1}^{n}\Gamma^{\beta_r}_{\alpha\beta}\,T^{\beta_1\cdots\beta_{r-1}\beta\beta_{r+1}\cdots\beta_n}_{\gamma_1\cdots\gamma_m} - \sum_{s=1}^{m}\Gamma^{\gamma}_{\alpha\gamma_s}\,T^{\beta_1\cdots\beta_n}_{\gamma_1\cdots\gamma_{s-1}\gamma\gamma_{s+1}\cdots\gamma_m}\,. \tag{21}$$

(See eq. (10.26) 0n page 304 of [3]). The use of the covariant derivative is important because the Ricci Tensor and the curvature satisfy the identity
$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu}\right) = 0$, so that from $(1)_1$ it follows $\nabla_\mu T^{\mu\nu} = 0$; this fact suggests that all the balance equations for the polyatomic gas must be expressed with the covariant derivative. Moreover, since we have done some changes of 4-dimensional coordinates, it is necessary that we use a derivative which does't depend on these changes, as the covariant derivative. More than that, we can use the following theorem

THEOREM 2: The following set of conditions are equivalent:

$$\begin{cases} \nabla_\alpha T^{\alpha\beta} = 0\,, \\ R^{\alpha\beta} - \frac{R}{2}\,g^{\alpha\beta} = \frac{8\,\pi\,G}{c^4}\,T^{\alpha\beta}\,, \end{cases} \Leftrightarrow \begin{cases} \nabla_\alpha T^{\alpha\beta} = 0\,, \\ R^{0\beta} - \frac{R}{2}\,g^{0\beta} = \frac{8\,\pi\,G}{c^4}\,T^{0\beta} \qquad \text{only as boundary conditions,} \\ R^{ab} - \frac{R}{2}\,g^{ab} = \frac{8\,\pi\,G}{c^4}\,T^{ab}\,. \end{cases} \tag{22}$$

PROOF: The implication $\Rightarrow$ is obvious, To prove $\Leftarrow$ we note that from the identity $\nabla_\alpha \left(R^{\alpha\beta} - \frac{R}{2}\, g^{\alpha\beta}\right) = 0$ and from the first equation in the right hand side of (22) it follows $\nabla_\alpha \left(R^{\alpha\beta} - \frac{R}{2}\, g^{\alpha\beta} - \frac{8\,\pi\, G}{c^4}\, T^{\alpha\beta}\right) = 0$, i.e.,

$$\partial_0 \left(R^{0\beta} - \frac{R}{2}\, g^{0\beta} - \frac{8\,\pi\, G}{c^4}\, T^{0\beta}\right) + \partial_a \left(R^{a\beta} - \frac{R}{2}\, g^{a\beta} - \frac{8\,\pi\, G}{c^4}\, T^{a\beta}\right) + \\ + \Gamma^\alpha_{\alpha\delta} \left(R^{\delta\beta} - \frac{R}{2}\, g^{\delta\beta} - \frac{8\,\pi\, G}{c^4}\, T^{\delta\beta}\right) + \Gamma^\beta_{\mu\nu} \left(R^{\mu\nu} - \frac{R}{2}\, g^{\mu\nu} - \frac{8\,\pi\, G}{c^4}\, T^{\mu\nu}\right) = 0\,.$$

For $\beta = b$ and for $\beta = 0$ these equations, by using the third equation in the right hand side of (22), become respectively

$$\partial_0 \left(R^{0b} - \frac{R}{2}\, g^{0b} - \frac{8\,\pi\, G}{c^4}\, T^{0b}\right) + \Gamma^\alpha_{\alpha 0} \left(R^{0b} - \frac{R}{2}\, g^{0b} - \frac{8\,\pi\, G}{c^4}\, T^{0b}\right) + \\ + 2\,\Gamma^b_{m0} \left(R^{m0} - \frac{R}{2}\, g^{m0} - \frac{8\,\pi\, G}{c^4}\, T^{m0}\right) + \Gamma^b_{00} \left(R^{00} - \frac{R}{2}\, g^{00} - \frac{8\,\pi\, G}{c^4}\, T^{00}\right) = 0\,,$$

$$\partial_0 \left(R^{00} - \frac{R}{2}\, g^{00} - \frac{8\,\pi\, G}{c^4}\, T^{00}\right) + \partial_a \left(R^{a0} - \frac{R}{2}\, g^{a0} - \frac{8\,\pi\, G}{c^4}\, T^{a0}\right) + \\ + \Gamma^\alpha_{\alpha\delta} \left(R^{\delta 0} - \frac{R}{2}\, g^{\delta 0} - \frac{8\,\pi\, G}{c^4}\, T^{\delta 0}\right) + \\ + \Gamma^0_{00} \left(R^{00} - \frac{R}{2}\, g^{00} - \frac{8\,\pi\, G}{c^4}\, T^{00}\right) + 2\,\Gamma^0_{m0} \left(R^{m0} - \frac{R}{2}\, g^{m0} - \frac{8\,\pi\, G}{c^4}\, T^{m0}\right) = 0\,.$$

It follows that $R^{00} - \frac{R}{2}\, g^{00} - \frac{8\,\pi\, G}{c^4}\, T^{00} = 0$, $R^{m0} - \frac{R}{2}\, g^{m0} - \frac{8\,\pi\, G}{c^4}\, T^{m0} = 0\ \forall\, x^0$ as consequence of the second equation in the right hand side of (22). Jointly with the first and third equation in the right hand side of (22), this result proves the left hand side of (22).

This result is important because it shows that we must impose only the equations

$$\nabla_\alpha\, V^\alpha = 0\,, \quad \nabla_\alpha\, T^{\alpha\beta} = 0\,, \quad \nabla_\alpha\, A^{\alpha\beta_1 \cdots \beta_n} = I^{\beta_1 \cdots \beta_n}\,, \quad \text{for} \quad n = 2\,, \cdots, N\,, \\ R^{ab} - \frac{R}{2}\, g^{ab} = \frac{8\,\pi\, G}{c^4}\, T^{ab}\,, \quad \text{for}\, a, b = 1, 2, 3. \tag{23}$$

The remaining equations of $(1)_1$ must be imposed only as initial conditions. This is in according with the general theory of constrained hyperbolic systems which have been considered in [5]-[7].

For the sequel we remark that $(23)_4$ is equivalent to $R_{ab} - \frac{R}{2}\, g_{ab} = \frac{8\,\pi\, G}{c^4}\, T_{ab}$ because in our metric we have $g_{0j} = 0$. We note also that in [1] (which used armonic coordinates and didn't concern polyatomic gases) there were some equations to be leaved out and the remaining one constitute an hyperbolic system. But it wasn't proved that the equations leaved out are consequences of the remaining ones and of the boundary conditions. We don't consider the possibility of exploiting this possibility because it is outside the scope of the present article.

In the present case, eq. $(23)_4$ are

$$\begin{aligned}
&\frac{G_2}{G_3}\,\partial_{00}\,G_3\,+F=\frac{8\,\pi\,G}{c^4}\,T_{11}\,,\\
&\frac{G_3}{2\,G_2}\,\partial_{00}\,G_2\,+\,\frac{1}{2}\,\partial_{00}\,G_3\,+\,\frac{1}{2\,G_2}\,\partial_{11}\,G_3\,+G=\frac{8\,\pi\,G}{c^4}\,T_{22}\,,\\
&T_{33}=T_{22}\,\sin^2\vartheta\,,\quad T_{12}=0\,,\quad T_{13}=0\,,\quad T_{23}=0\,,
\end{aligned} \tag{24}$$

where F and G are explicit functions of G_2, G_3 and of their first order derivatives with respect to x^0 and s. The first two of these equations allow to determinate G_2 and G_3; the other are constraints on the energy-momentum tensor, but they are automatically satisfied because of the requirement of isotropy of the universe. As we have assumed that $g_{\alpha\beta}$ has the decomposition (2) in the initial 4-dimensional variables of the reference comoving with the fluid, then the same thing must be said for all the other quantities. For example, $T^{\alpha\beta}$ must have the decomposition

$$T_{\alpha\beta}=\begin{pmatrix} e\left(x^0\,,\,s\right) & \frac{q\left(x^0\,,\,s\right)}{c}\,\frac{x_j}{s} \\ \frac{q\left(x^0\,,\,s\right)}{c}\,\frac{x_i}{s} & t_1\left(x^0\,,\,s\right)\,\frac{x_i\,x_j}{s^2}\,+\,t_2\left(x^0\,,\,s\right)\,\delta_{ij}\end{pmatrix}\,,$$

with

$$\frac{1}{3}\,\left(t_1+t_2\right)\left(g_2+g_3\right)+\,\frac{2}{3}\,t_2\,g_3=p+\Pi\,.$$

(Definition of pressure and the dynamic pressure).

To see briefly how it becomes after the two changes of 4-dimensional variables, we may consider the quadratic form

$$T_{\alpha\beta}\,d\,x^\alpha\,d\,x^\beta=T_{\alpha\beta}\,\frac{\partial\,x^\alpha}{\partial\,X^\mu}\,\frac{\partial\,x^\beta}{\partial\,X^\nu}\,d\,X^\mu d\,X^\nu=\overset{N}{T}_{\mu\nu}\,d\,X^\mu d\,X^\nu\,,$$

where in the right hand side we have used a generic change of 4-dimensional variables; so, by expressing this quadratic form in the new 4-dimensional coordinates, then $\overset{N}{T}_{\mu\nu}$ comes out automatically as the associated matrix. In particular, after our two changes of 4-dimensional coordinates the energy momentum tensor takes the form

$$T_{\alpha\beta}=\begin{pmatrix} T_{00} & T_{01} & 0 & 0\\ T_{01} & T_{11} & 0 & 0\\ 0 & 0 & T_{22} & 0\\ 0 & 0 & & T_{22}\sin^2\vartheta\end{pmatrix}\,, \tag{25}$$

with explicit expressions of T_{00}, T_{01}, T_{11}, T_{22} which we don't report. So the conditions $(24)_{3-6}$ are automatically satisfied. In similar way can be treated all the other tensors appearing in the closure. But they aren't necessary here because they will come out automatically when we express them in terms of physical variables. These have been found in [8]-[10] but only in the case of a Minkowsky metric; in the next section we will find what changes with the present metric.

4 The closure of the balance equations with the present metric

We adopt for $A^{\alpha_1\cdots\alpha_{n+1}}$, $I^{\alpha_1\cdots\alpha_n}$ the expressions which has been found in [8]-[10], i.e.,

$$A^{\alpha_1\cdots\alpha_{n+1}} = \frac{c}{m^{n-1}} \int_{\Re^3} \int_0^{+\infty} f\, p^{\alpha_1} \cdots p^{\alpha_{n+1}} \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^n \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,$$
$$I^{\alpha_1\cdots\alpha_n} = \frac{c}{m^{n-1}} \int_{\Re^3} \int_0^{+\infty} Q\, p^{\alpha_1} \cdots p^{\alpha_n} \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^n \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,, \qquad (26)$$

with the distribution function f given by

$$f = e^{-1-\frac{\chi}{k_B}}\,, \quad \chi = \sum_{n=0}^{N} \lambda_{\alpha_1\alpha_2\cdots\alpha_n} p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_n} \frac{1}{m^{n-1}} \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^n\,, \qquad (27)$$

where k_B is the Boltzmann constant, m is the particle mass, $\mathcal{I}$ is the contribution to energy from internal modes and $\lambda_{\alpha_1\cdots\alpha_n}$ are Lagrange multipliers which are taken as independent variables. The expression $(26)_2$ was found in [11]. To express every thing in terms of physical variables we need an inversion of variables; this was realized in [8]-[10] but by considering a Minkowsky metric and by calculating the integrals in the reference frame comoving with the fluid. In the present approach, the metric $g_{\alpha\beta}$ is an unknown to be determined; moreover, in the reference frame comoving with the fluid the metric was given by (2) but, after that, we made the two changes of 4-dimensional variables (4) and that which uses polar coordinates. So, let us see the implications of these 2 changes. To this end, let us recall from literature that under a change of 4-dimensional variables the left hand side of Einstein Equation transforms according to the law

$$\overset{N}{R}{}^{\mu\nu} - \frac{1}{2} \overset{N}{R} \overset{N}{g}{}^{\mu\nu} = \left(R^{\alpha\beta} - \frac{1}{2} R\, g^{\alpha\beta}\right) \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} \quad \to \quad \overset{N}{T}{}^{\mu\nu} = T^{\alpha\beta} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta}\,,$$

where the quantities $\overset{N}{(\cdots)}$ denote the expression of $(\cdots)$ after the change of the 4-dimensional variables; moreover, the last implication is a consequence of $(1)_1$.
The same property must hold also for the other tensors $A^{\alpha_1\cdots\alpha_{n+1}}$, $I^{\alpha_1\cdots\alpha_n}$. In particular, if in the initial comoving reference frame we have $\overset{H}{V}_\alpha \equiv (\rho\, c\,, 0\,, 0\,, 0)$, then

$$\overset{H}{V}_\alpha\, d\,x^\alpha = \rho\, c\, d\,x^0 = \rho\, c \left(\frac{\partial f}{\partial X^0} d\,X^0 + \frac{\partial f}{\partial S} d\,S\right) \quad \to \quad \overset{N}{V}_\alpha \equiv \rho\, c \left(\frac{\partial f}{\partial X^0}\,, \frac{\partial f}{\partial S}\,, 0\,, 0\right).$$

From this relation we obtain

$$\overset{N}{V}_\alpha \overset{N}{V}_\beta\, g^{\alpha\beta} = \rho^2\, c^2 \left[\left(\frac{\partial f}{\partial X^0}\right)^2 + \frac{1}{G_2} \left(\frac{\partial f}{\partial s}\right)^2\right] \to \quad .$$

$$\overset{N}{\rho} = \rho \sqrt{\left(\frac{\partial f}{\partial X^0}\right)^2 + \frac{1}{G_2}\left(\frac{\partial f}{\partial S}\right)^2}\,, \quad \overset{N}{U}_\alpha \equiv \frac{c}{\sqrt{\left(\frac{\partial f}{\partial X^0}\right)^2 + \frac{1}{G_2}\left(\frac{\partial f}{\partial S}\right)^2}} \left(\frac{\partial f}{\partial X^0}\,, \frac{\partial f}{\partial S}\,, 0\,, 0\right).$$

This relations show that, altough ρ seems apparently a scalar function, it isn't invariant and in our last 4-dimensional coordinates we are no more in a reference frame comoving with the fluid. But we can define v from

$$\frac{v}{\sqrt{1+\frac{v^2}{c^2G_2}}}=\frac{c\,\frac{\partial f}{\partial S}}{\sqrt{\left(\frac{\partial f}{\partial X^0}\right)^2+\frac{1}{G_2}\left(\frac{\partial f}{\partial S}\right)^2}}\to\frac{\frac{\partial f}{\partial X^0}}{\sqrt{\left(\frac{\partial f}{\partial X^0}\right)^2+\frac{1}{G_2}\left(\frac{\partial f}{\partial S}\right)^2}}=\frac{1}{\sqrt{1+\frac{v^2}{c^2G_2}}}\,. \tag{28}$$

By calling $\left(1+\frac{v^2}{c^2G_2}\right)^{-1/2}=\Gamma(v)$, we have obtained that

$$\overset{N}{U}_\alpha\equiv\Gamma(v)\;(c\,,\,v\,,\,0\,,\,0)\;,\quad \overset{N}{U}{}^\alpha\equiv\Gamma(v)\left(c\,,\,\frac{v}{G_2}\,,\,0\,,\,0\right)\,,\quad \overset{N}{V}_\alpha=\overset{N}{\rho}\,\overset{N}{U}_\alpha\;.$$

Similarly, equilibrium (denoted with the apex E) is defined as the state where $\lambda=\lambda^E$, $\lambda_\alpha=\lambda^E_\alpha$, $\lambda_{\alpha_1\cdots\alpha_n}=0$ for $n=2,\cdots,N$ and we have $T^{\alpha\beta}_E=\frac{e+p}{\rho^2c^2}V^\alpha V^\beta-p\,g^{\alpha\beta}$. It follows that

$$\rho^2e=T^{\alpha\beta}_E\frac{V_\alpha V_\beta}{c^2}\,,\,p=\frac{1}{3}\left(e\,-\,T^{\alpha\beta}_E g_{\alpha\beta}\right)\,,\to\overset{N}{\rho}{}^2\,\overset{N}{e}=\rho^2e\,,$$

$$\overset{N}{e}=e\left(\frac{\rho}{\overset{N}{\rho}}\right)^2=\frac{e}{\left(\frac{\partial f}{\partial X^0}\right)^2+\frac{1}{G_2}\left(\frac{\partial f}{\partial S}\right)^2}\,,\,\overset{N}{p}=\frac{1}{3}\left(\overset{N}{e}\,-e+3p\right)=p+\frac{\overset{N}{e}\,-e}{3}\,.$$

So also the energy density e and the pressure p aren't invariant. More completely, at equilibrium we have still (25) but with

$$T^E_{00}=(e+p)\,\Gamma^2(v)-\,p\,,\quad T^E_{01}=(e+p)\,\Gamma^2(v)\,\frac{v}{c}\,,\quad T^E_{11}=\Gamma^2(v)\left(e\,\frac{v^2}{c^2}\,-\,p\,G_2\right)\,,$$
$$T^E_{22}=-p\,G_3\,.$$

Now we have to calculate again the expressions of e and p and not to simply use the results of [8]-[10]. In what follows we will operate only in our last 4-dimensional variables so that every thing will be referred to them and we will omit both the apex $\overset{N}{(\cdots)}$. But it will be necessary to use the decomposition $U_\alpha\equiv\Gamma(v)\;(c\,,\,v\,,\,0\,,\,0)$, $\quad V_\alpha=\rho\,U_\alpha$. For the sequel it will be useful to calculate also

$$A_E^{\alpha_1\cdots\alpha_{n+1}}=\frac{c}{m^{n-1}}\int_{\Re^3}\int_0^{+\infty}f_E\,p^{\alpha_1}\,\cdots\,p^{\alpha_{n+1}}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^n\varphi(\mathcal{I})\,d\mathcal{I}\,d\vec{P}\,, \tag{29}$$

where

$$f_E=e^{-1-\frac{\chi_E}{k_B}}\,,\quad \chi_E=m\,\lambda^E+\lambda^E_\mu p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\,.$$

It follows that

$$d\,A_E^{\alpha_1\cdots\alpha_{n+1}}=-\,\frac{m}{k_B}\left(A_E^{\alpha_1\cdots\alpha_{n+1}}d\,\lambda^E\,+\,A_E^{\alpha_1\cdots\alpha_{n+2}}d\,\lambda^E_{\alpha_{n+2}}\right)\,. \tag{30}$$

This equation, written for $n=0$ is

$$d\left(\rho U^{\alpha_1}\right) = -\frac{m}{k_B}\left[\rho U^{\alpha_1}\, d\lambda^E + \left(e\,\frac{U^{\alpha_1}U^{\alpha_2}}{c^2} + p\, h^{\alpha_1\alpha_2}\right) d\lambda^E_{\alpha_2}\right],$$

whose contraction with U_{α_1} allows to determine

$$d\lambda^E = -\frac{k_B}{m\,\rho}\, d\rho - \frac{e}{\rho\, c^2}\, U^{\alpha_2}\, d\lambda^E_{\alpha_2}\,.$$

By substituting this in eq. (30), we find

$$d A_E^{\alpha_1\cdots\alpha_{n+1}} = A_E^{\alpha_1\cdots\alpha_{n+1}}\left(\frac{1}{\rho}\, d\rho + \frac{e\,m}{\rho\, c^2 k_B}\, U^\gamma\, d\lambda^E_\gamma\right) - \frac{m}{k_B}\, A_E^{\alpha_1\cdots\alpha_{n+2}} d\lambda^E_{\alpha_{n+2}}\,.$$

If we take ρ and λ^E_γ as independent variables, the coefficient of $d\rho$ shows that $A_E^{\alpha_1\cdots\alpha_{n+1}}$ is linear and homogeneos in the variable ρ, while the coefficient of $d\lambda^E_\gamma$ allows to determine

$$A_E^{\alpha_1\cdots\alpha_{n+2}} = -\frac{k_B}{m}\,\frac{\partial A_E^{\alpha_1\cdots\alpha_{n+1}}}{\partial \lambda^E_{\alpha_{n+2}}} + \frac{e}{\rho\, c^2}\, A_E^{\alpha_1\cdots\alpha_{n+1}}\, U^{\alpha_{n+2}}\,. \tag{31}$$

Thanks to this result, all the tensors $A_E^{\alpha_1\cdots\alpha_{n+1}}$ are determined in terms of the previous ones. Obviously, we must be careful and express everything in terms of ρ and λ^E_γ. Regarding the second one of these, we note that

$$\lambda^E_\gamma = \frac{U_\gamma}{T} \quad\rightarrow\quad T = \frac{c}{\sqrt{\lambda^E_\delta \lambda^{E\delta}}}\,, \quad U_\gamma = \frac{c}{\sqrt{\lambda^E_\delta \lambda^{E\delta}}}\,\lambda^E_\gamma\,, \quad A^\gamma_E = \rho\, U^\gamma = \frac{\rho\, c}{\sqrt{\lambda^E_\delta \lambda^{E\delta}}}\,\lambda^E_\gamma\,.$$

As a test, let us consider eq. (31) for $n=0$ and use the projector $h^{\alpha\beta} = -\,g^{\alpha\beta} + \frac{U^\alpha U^\beta}{c^2}$, i.e.,

$$T_E^{\alpha_1\alpha_2} = -\frac{k_B}{m}\,\frac{\partial A^{\alpha_1}}{\partial \lambda^E_{\alpha_2}} + \frac{e}{\rho\, c^2}\, A_E^{\alpha_1}\, U^{\alpha_2} == -\frac{k_B}{m}\left(\frac{\rho\, c}{\sqrt{\lambda^E_\delta \lambda^{E\delta}}} g^{\alpha_1\alpha_2} - \frac{\rho\, c}{\left(\lambda^E_\delta \lambda^{E\delta}\right)^{3/2}}\lambda_E^{\alpha_1}\,\lambda_E^{\alpha_2}\right) + \\ + \frac{e}{\lambda^E_\delta \lambda^{E\delta}}\lambda_E^{\alpha_1}\,\lambda_E^{\alpha_2} = \frac{k_B}{m}\,\rho\, T\, h^{\alpha_1\alpha_2} + \frac{e}{c^2}\, U^{\alpha_1} U^{\alpha_2}\,.$$

So we have obtained the correct expression for the coefficient of $U^{\alpha_1}U^{\alpha_2}$, while the other term gives

$$p = \frac{k_B}{m}\,\rho\, T = \rho\,\frac{c^2}{\gamma} \quad \text{(as in the case with a Minkowsky metric), with}\, \gamma = \frac{m\, c^2}{k_B T}\,. \tag{32}$$

We note that (31) doesn't permit to obtain the expression of the energy e; so to find it we must go back to the definition $(26)_1$ for $n=0$ and contracted by U_{α_1} and for $n=1$ and contracted by $U_{\alpha_1}U_{\alpha_2}$; the effective calculations are performed in appendix and the result is

$$\frac{e}{p\,c^2} = \Gamma(v)\left[\int_0^{+\infty}\int_0^{+\infty} e^{-\gamma\Gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\cosh\, s}\sinh^2 s\left(\cosh\, s\,\frac{\sinh\,\xi}{\xi}\;+\right.\right.$$
$$\left.\left.+\frac{1}{\gamma\Gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)}\,\frac{\sinh\,\xi-\xi\,\cosh\,\xi}{\xi}\right)\varphi(\mathcal{I})\,d\,s\,d\mathcal{I}\right]^{-1}\cdot$$
$$\cdot\int_0^{+\infty}\int_0^{+\infty} e^{-\gamma\Gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\cosh\, s}\sinh^2 s\left(\cosh^2\, s\,\frac{\sinh\,\xi}{\xi}\;+\right.$$
$$\left.+\;\frac{\xi\,\sinh\,\xi+2\left[1+\gamma\Gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\cosh\, s\right]\frac{\sinh\,\xi-\xi\,\cosh\,\xi}{\xi}}{\gamma^2\Gamma^2\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^2}\right)\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\varphi(\mathcal{I})\,d\,s\,d\mathcal{I}\,, \tag{33}$$

where

$$\xi = \frac{\gamma\,\Gamma}{\sqrt{-\,G_2}}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\sinh\, s\,\frac{v}{c}\,.$$

From these relations it follows

$$\lim_{v\to 0}\xi = 0\,,$$
$$\lim_{v\to 0}\frac{e}{p\,c^2} = \frac{\int_0^{+\infty}\int_0^{+\infty} e^{-\gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\cosh\, s}\sinh^2 s\,\cosh^2\, s\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\varphi(\mathcal{I})\,d\,s\,d\mathcal{I}}{\int_0^{+\infty}\int_0^{+\infty} e^{-\gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\cosh\, s}\sinh^2 s\cosh\, s\,\varphi(\mathcal{I})\,d\,s\,d\mathcal{I}}\,,$$

as in the case [9] without the gravitational field.
We note that (31) holds also in the case without gravitational field (the passages here followed don't take into account this presence), so the expression of $A_E^{\alpha_1\cdots\alpha_{n+1}}$ is the same. The only difference is the expression of e that now replaces that in eqs. $(12)_{2,3}$ of [9] and $(3)_2$ of [10]. So we obtain the expressions (14),(16) of [9].
We have now to explicitate eq. (25) according to the definitions

$$0 = U_\alpha q^\alpha \quad\to\quad q^0 = -\,\frac{v}{c}\,q\,,$$
$$U_\alpha T^{\alpha\beta} = e\,U^\beta\;+\;q^\beta \quad\to\quad q^\alpha = -h^\alpha_\mu\,U_\beta\,T^{\beta\mu}\,,$$
$$h_{\alpha\beta}\,T^{\alpha\beta} = 3\,(p+\Pi)\,,\quad t^{<\mu\nu>_3} = \left(h^\mu_\alpha\,h^\nu\beta\;-\;\frac{1}{3}\,h_{\alpha\beta}h^{\mu\nu}\right)T^{\alpha\beta}\,,$$

from which it follows

$$T^{\mu\nu} = e\,\frac{U^\mu U^\nu}{c^2} + (p+\Pi)h^{\mu\nu} + \frac{2}{c^2}U^{(\mu}q^{\nu)}\;+\;t^{<\mu\nu>_3}\,.$$

We find

$$T^{\alpha\beta} = \tag{34}$$

$$= \begin{pmatrix} \begin{matrix} e\,\Gamma^2 + (p+\Pi)\left(\Gamma^2-1\right) \\ -2\Gamma\frac{v}{c^2}\,q \\ -2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v^2}{c^2}\,t^{<>} \end{matrix} & \begin{matrix} (e+p+\Pi)\Gamma^2\frac{v}{c\,G_2} + \\ +\frac{q}{c}\,\Gamma\left(1-\frac{v^2}{c^2G_2}\right) + \\ +2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v}{c}\,t^{<>} \end{matrix} & 0 & 0 \\ \begin{matrix} (e+p+\Pi)\Gamma^2\frac{v}{c\,G_2} + \\ +\frac{q}{c}\,\Gamma\left(1-\frac{v^2}{c^2G_2}\right) + \\ +2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v}{c}\,t^{<>} \end{matrix} & \begin{matrix} e\,\frac{\Gamma^2v^2}{c^2\,(G_2)^2} - (p+\Pi)\,\frac{\Gamma^2}{G_2} + \\ +2\,q\,\frac{\Gamma\,v}{c^2G_2} \\ -2\,\frac{G_3}{G_2}\,\Gamma^2 t^{<>} \end{matrix} & 0 & 0 \\ 0 & 0 & t^{<>} - \frac{p+\Pi}{G_3} & 0 \\ 0 & 0 & & \frac{t^{<>} - \frac{p+\Pi}{G_3}}{\sin^2\vartheta} \end{pmatrix},$$

$$t^{<\alpha\beta>_3} = \begin{pmatrix} -2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v^2}{c^2}\,t^{<>} & +2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v}{c}\,t^{<>} & 0 & 0 \\ +2\,\frac{G_3}{G_2}\,\Gamma^2\frac{v}{c}\,t^{<>} & -2\,\frac{G_3}{G_2}\,\Gamma^2 t^{<>} & 0 & 0 \\ 0 & 0 & t^{<>} & 0 \\ 0 & 0 & & \frac{t^{<>}}{\sin^2\vartheta} \end{pmatrix}.$$

With these steps we have introduced 5 new independent variables, n, γ, U_α. To eliminate this drawback we have to consider the system $0 = V^\alpha - V^\alpha_E$, $0 = U_\alpha U_\beta \left(T^{\alpha\beta} - T^{\alpha\beta}_E\right)$ to obtain $\lambda - \lambda^E$, $\lambda_\beta - \lambda^E_\beta$ which now are substituted by the 5 new independent variables. Another possibility is to use only physical variables and, in this case, all the Lagrange multipliers must be expressed in terms of them. For the sake of simplicity, I consider here only the 15 moments model for polyatomic gases as in [9]. In this case we have to consider the system constituted by eqs. $(26)_{1,2}$ and $(26)_3$ contracted by $U_\alpha U_\beta U_\gamma$ of [9], with Δ defined in (23) of [9], i.e, $\Delta = \frac{4}{c^2} U_\alpha U_\beta U_\gamma \left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_E\right)$. All the other steps in [9] hold also in the present case and we find their same closure (35), i.e.,

$$\begin{aligned} A^{\alpha\beta\gamma} = &\left(\rho\,\theta_{02} + \frac{1}{4c^4}\,\Delta\right) U^\alpha U^\beta U^\gamma + \left(\rho\,c^2\,\theta_{12} - \frac{3}{4c^2}\,\frac{N^\Delta}{D_4}\,\Delta - 3\,\frac{N^\Pi}{D_4}\,\Pi\right) h^{(\alpha\beta}U^{\gamma)} \\ &+ \frac{3}{c^2}\frac{N_3}{D_3}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{3}{5}\frac{N_{31}}{D_3} h^{(\alpha\beta}q^{\gamma)} + 3C_5 t^{(<\alpha\beta>_3}U^{\gamma)}\,, \end{aligned} \tag{35}$$

$$\begin{aligned} I^{\beta\gamma} = &- \frac{1}{4c^4\tau}\,\Delta\,U^\beta U^\gamma + \frac{1}{4c^2\tau}\,\frac{N^\Delta}{D_4}\,\Delta\,h^{\beta\gamma} + \frac{1}{\tau}\,\frac{N^\Pi}{D_4}\,\Pi\,h^{\beta\gamma} + \\ &+ \frac{1}{c^2\tau}\left(\frac{\theta_{1,3}}{\theta_{1,2}} - 2\frac{N_3}{D_3}\right) U^{(\beta}\,q^{\gamma)} + - \frac{1}{\tau}\,C_5\,t^{<\beta\gamma>_3}\,. \end{aligned}$$

where all the coefficients are reported in [9].

5 Hyperbolicity of the equations (23) with metric (19)

Let us define $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \frac{8\pi G}{c^4} T_{\mu\nu}$; in this way eqs. $(23)_4$ become $S_{ab} = 0$. We prove now the following
THEOREM 3: "The statement $S_{ab} = 0$ with $a = b = 1, 2, 3$, coupled with initial conditions including

$$g_3 - g_2 \sin^2 \vartheta = 0\,,\ \partial_0 \left(g_3 - g_2 \sin^2 \vartheta\right) = 0\,,\ \partial_2\, g_1 = 0\,,\ \partial_3\, g_1 = 0\,,\ \partial_2\, g_2 = 0\,,\qquad (36)$$
$$\partial_3\, g_2 = 0\,,\ \partial_{02}\, g_1 = 0\,,\ \partial_{03}\, g_1 = 0\,,\ \partial_{02}\, g_2 = 0\,,\ \partial_{03}\, g_2 = 0\,,$$

implies that, also outside of the initial manifold, we have

$$g_3 - g_2 \sin^2 \vartheta = 0\,,\ \partial_2\, g_1 = 0\,,\ \partial_3\, g_1 = 0\,,\ \partial_2\, g_2 = 0\,,\ \partial_3\, g_2 = 0\,,\ S_{ab} = 0 \text{ also with } a \neq b\text{"}. \qquad (37)$$

To prove this theorem it suffices to verify that it is effectiveely a solution of $S_{ab} = 0$ with $ab = 11, 12, 13, 22, 23, 33$. This is true because for the solution satisfying $(37)_{1-5}$ we have

$$\begin{cases} S_{ab} = 0 \quad \text{if} \quad a \neq b \\ S_{33} = S_{22} \sin^2 \vartheta\,, \end{cases} \qquad \text{identically, for the expression } (18)_4 \text{ and similar; also for } (25)\,,$$

while $S_{11} = 0$ and $S_{22} = 0$ become equations for the determination of g_1 and g_2.
CONSEQUENCES: For the study of hyperbolicity it suffices to consider only the equations $(23)_{1-3}$ and the equations $(23)_4$ only for $a = b = 1, 2, 3$. Moreover, we have to include (36) but only as initial conditions in the initial manifold.
We see that eqs. $(23)_4$ for $a = b$ are partial differential equations of the second order, while $(23)_{1-3}$ are of first order. To reduce to a system all of the first order, we define

$$\partial_\alpha\, g_a = g_a^\alpha\,, \quad \to \quad \partial_{[\beta}\, g_a^{\alpha]} = 0\,.$$

Here the equations on the right hand side are the integrability conditions on those in the left hand side. In this way the system $(23)_{1-3}$ and $(23)_4$ with $a = b$ can be written as

$$\begin{aligned}
&\partial_\alpha V^\alpha = H\,,\ \partial_\alpha T^{\alpha\beta} = H^\beta\,,\ \partial_\alpha A^{\alpha\beta\gamma} = H^{\beta\gamma}\,,\ \partial_0\, g_a = g_a^0\,,\ \partial_0\, g_a^b - \partial_b\, g_a^0 = 0\,,\\
&\frac{g_1}{2\, g_2}\, \partial_0\, g_2^0 + \frac{g_1}{2\, g_3}\, \partial_0\, g_3^0 + \frac{g_1}{2\, g_2 g_3}\, \partial_2\, g_3^2 + \frac{g_1}{2\, g_2 g_3}\, \partial_3\, g_2^3 = K^1\\
&\frac{g_2}{2\, g_1}\, \partial_0\, g_1^0 + \frac{g_2}{2\, g_3}\, \partial_0\, g_3^0 + \frac{g_2}{2\, g_1 g_3}\, \partial_1\, g_3^1 + \frac{g_2}{2\, g_1 g_3}\, \partial_3\, g_1^3 = K^2\,,\\
&\frac{g_3}{2\, g_2}\, \partial_0\, g_2^0 + \frac{g_3}{2\, g_1}\, \partial_0\, g_1^0 + \frac{g_3}{2\, g_1 g_2}\, \partial_2\, g_1^2 + \frac{g_3}{2\, g_1 g_2}\, \partial_1\, g_2^1 = K^3\,,
\end{aligned} \qquad (38)$$

where H, H^β, $H^{\beta\gamma}$, K^1, K^2 and K^3 are functions of $\rho\,,\ \gamma\,,\ U^\alpha\,,\ \Pi\,,\ q^\alpha\,,\ t_3^{<\alpha\beta>}$, g_a^α.
The equations to study wave propagation can be obtained by substituting ∂_0 with $-u\,d$ and ∂_k with $n_k\, d$ an unitary vector; the remaining part of each equation has to be put

equal to zero. In our case these equations are

$$
\begin{aligned}
& -\, u\, d V^0 + n_k\, d V^k = 0\,, \; -\, u\, d T^{0\beta} + n_k\, d T^{k\beta} = 0\,, \quad -\, u\, d A^{0\beta\gamma} + n_k\, d A^{k\beta\gamma} = 0\,, \\
& -\, u\, d g_a = 0\,, \quad -\, u\, d g^b_a - n_b\, d g^0_a = 0\,, \\
& -\, u \left(\frac{g_1}{2\, g_2}\, d g^0_2 + \frac{g_1}{2\, g_3}\, d g^0_3 \right) + \frac{g_1}{2\, g_2 g_3} n_2\, d g^2_3 + \frac{g_1}{2\, g_2 g_3}\, n_3\, d g^3_2 = 0 \\
& -\, u \left(\frac{g_2}{2\, g_1}\, d g^0_1 + \frac{g_2}{2\, g_3}\, d g^0_3 \right) + \frac{g_2}{2\, g_1 g_3}\, n_1\, d g^1_3 + \frac{g_2}{2\, g_1 g_3}\, n_3\, d g^3_1 = 0\,, \\
& -\, u \left(\frac{g_3}{2\, g_2}\, d g^0_2 + \frac{g_3}{2\, g_1}\, d g^0_1 \right) + \frac{g_3}{2\, g_1 g_2}\, n_2\, d g^2_1 + \frac{g_3}{2\, g_1 g_2}\, n_1\, d g^1_2 = 0\,.
\end{aligned}
\tag{39}
$$

We note that the last 5 equations don't depend on the other variables which are present in $(39)_{1-3}$. They depend only on the 15 variables $d\, g_a$, $d\, g^\alpha_a$.
If $u = 0$, the equations $(39)_{4-5}$ give $d\, g^0_a = 0$; jointly with $(39)_{6-8}$ they give 6 constraints on the above mentioned 15 variables and we conclude that the eigenvalue $u = 0$ has at least multiplicity 9.
If $u \neq 0$, the equations $(39)_{4-5}$ give $d\, g_a = 0$, $d\, g^b_a = -\frac{n_b}{u}\, d\, g^0_a$. By substituting in $(39)_{6-8}$ we obtain the system

$$
\begin{pmatrix}
0 & u^2 \frac{g_1}{2\, g_2} + (n_3)^2\, \frac{g_1}{2\, g_2\, g_3} & u^2 \frac{g_1}{2\, g_3} + (n_2)^2\, \frac{g_1}{2\, g_2\, g_3} \\
u^2 \frac{g_2}{2\, g_1} + (n_3)^2\, \frac{g_2}{2\, g_1\, g_3} & 0 & u^2 \frac{g_2}{2\, g_3} + (n_1)^2\, \frac{g_2}{2\, g_1\, g_3} \\
u^2 \frac{g_3}{2\, g_1} + (n_2)^2\, \frac{g_3}{2\, g_1\, g_2} & u^2 \frac{g_3}{2\, g_2} + (n_1)^2\, \frac{g_3}{2\, g_1\, g_2} & 0
\end{pmatrix}
\begin{pmatrix} d\, g^0_1 \\ d\, g^0_2 \\ d\, g^0_3 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

So we obtain 6 other independent eigenvectors correspondind to the 6 real eigenvalues which are the solutions of

$$
u^2 = -\, \frac{(n_1)^2}{g_1}\,, \quad u^2 = -\, \frac{(n_2)^2}{g_2}\,, \quad u^2 = -\, \frac{(n_3)^2}{g_3}\,.
$$

Since the sum ot the independent eigenvectors is 15, the hyperbolicity is proved for $(39)_{4-8}$. Regarding the role of $(39)_{1-3}$, we can take for them $d\, g_a = 0$, $d\, g^\alpha_a = 0$; in other words, they are the same of the case with $g_{\alpha\beta}$ constant. So we can use the results of [9] and say that they give other 15 linearly independent eigenvectors, So the hyperbolicity is proved for all the set (39).
We note that, if we start from the metric (16) and do the same calculations, we find that the hyperbolicity requirement isn't satisfied. This is not a problem because, by applying the results of [5], [6] and [7], we find that the metric (16) is a consequence of the metric here used $g_{\alpha\beta} = diag\ (1\,, g_1\,, g_2\,, g_3)$ (see (19)) and of suitable initial conditions.

6 The Friedmann-Robertson-Walker (FRW) metric

Let us see if it is possible to obtain as a particular solution the FRW metric $g_{\alpha\beta} = diag\left(1\,,\,-\alpha\,g\,,\,-\alpha\,(x_1)^2\,,\,-\alpha\,(x_1)^2\,\sin^2 x^2\right)$ with $\alpha = \alpha\,(x_0)$, $g = \frac{1}{1-\epsilon\,(x^1)^2}$, $\alpha > 0$,

$$\epsilon = \begin{cases} 0 & \text{for flat space-time}\,, \\ 1 & \text{for a closed space-time} \\ -1 & \text{for open space-time}\,. \end{cases}$$

With this metric the components of Einstein Equations (1), $S_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2}\,g_{\alpha\beta} = \frac{8\,\pi\,G}{c^4}\,T_{\mu\nu}$, become

$$\begin{aligned}
&S_{00} = -\,\frac{3}{2}\,\frac{\alpha''}{\alpha} + \frac{1}{\alpha\,x^1}\,\frac{g'}{g^2} + \frac{1}{\alpha\,(x^1)^2} - \frac{1}{\alpha\,g\,(x^1)^2} = \frac{8\,\pi\,G}{c^4}\,T_{00}\,, \\
&S_{11} = -\,\frac{1}{2}\,\alpha''\,g + \frac{1}{2}\,g\,\frac{(\alpha')^2}{\alpha} - \frac{g}{(x^1)^2} + \frac{1}{(x^1)^2} = \frac{8\,\pi\,G}{c^4}\,T_{11}\,, \\
&S_{22} = \frac{(x^1)^2}{g}\,S_{11} + \frac{1}{2}\,x^1\left(\frac{1}{g} - 1\right)' - \left(\frac{1}{g} - 1\right) = \frac{8\,\pi\,G}{c^4}\,T_{22}\,, \\
&S_{33} = S_{22}\,\sin^2\vartheta = \frac{8\,\pi\,G}{c^4}\,T_{33}\,, \\
&S_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \to T_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta\,.
\end{aligned}$$

From the third one of these relations we see that $S_{22} = \frac{(x^1)^2}{g}\,S_{11}$ if and only if $\frac{1}{g} - 1$ satisfies the differential equation

$$\frac{1}{2}\,x^1\left(\frac{1}{g} - 1\right)' = \frac{1}{g} - 1 \quad \leftrightarrow \quad g = \frac{1}{1 - \epsilon\,(x^1)^2}\,,$$

with ϵ an arbitrary integration constant; so the above values of ϵ are particular cases with physical meaning.
By comparing with the expression (34) of $T_{\alpha\beta}$, we see that

- The result $T_{12} = 0$ calculated at equilibrium implies that $v = 0$. So a first initial condition necessary to have the FRW metric is that in the initial manifold we have $v = 0$; after that, for the hyperbolicity of the system we will have $v = 0$ also outside of it.
- The result $T_{12} = 0$ outside of equilibrium implies that $q = 0$, i.e., there is no heat flux.
- The result $S_{22} = \frac{(x^1)^2}{g}\,S_{11}$ implies that $0 = T_{22} - \frac{(x^1)^2}{g}\,T_{11} = 3\,\alpha^2\,\left(x^1\right)^4\,t^{<>}$, i.e., there is no viscous deviatoric stress $t^{<>}$ and we have

$$T^{\alpha\beta} = \text{diag}\left(e\,,\,\frac{p+\Pi}{\alpha\,g}\,,\,\frac{p+\Pi}{\alpha\,(x^1)^2}\,,\,\frac{p+\Pi}{\alpha\,(x^1)^2\,\sin^2\vartheta}\right).$$

So, $v = 0$, $q = 0$, $t^{<>} = 0$ aren't conditions dictated by physics, but only necessary conditions in order to have the FRW metric otherwise we will obtain another metric; physical experiments can only test if a particular polyatomic gas satisfies these conditions.

7 Conclusions

It has been proved here that, with the metric $g_{\alpha\beta} = \text{diag}\,(1\,,\, g_1\,,\, g_2\,,\, g_3)\,,\, g_a = g_a\,(x^\alpha)$, Eintein Equations coupled with that of a polyatomic gas constitute a set of hyperbolic equations. The conditions coming out from isotropy give a metric that is a particular case of the previous one, but in this way hyperbolicity is lost; it is recovered only if we study hyperbolicity with the metric $g_{\alpha\beta} = \text{diag}\,(1\,,\, g_1\,,\, g_2\,,\, g_3)$ and impose only as boundary conditions $g_3 = g_2 \sin^2\vartheta$, $g_1 = g_1\left(x^0\,,\, x^1\right)$, $g_2 = g_2\left(x^0\,,\, x^1\right)$.
Moreover, it has been found that only with the further boundary conditions $v = 0$, $q = 0$, $t^{<>} = 0$ we may obtain the Friedmann-Robertson-Walker metric.
Further investigations are possible; for example, the isotropy isn't imposed in the articles on polyatomic gases with the Minkowsky metric and there is no reason to impose it here. So the present article can be considered as a starting point to this further development. Moreover, the case can be studied where the gravitational field operates externally to the polyatomic gas, for example outside a black hole; this can be used to estimate the radius of the bkack hole by observing how the gases orbiting it move. Previous estimates not based on Extended Thermodynamics nor on polyatomic gases have given results very different from the "photo" that was made recently of one of these black holes.

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A Appendix: Integrations of $(29)_1$ to obtain the energy density e

First of all we have to clarify our ideas about the integral factor $d\vec{P}$ because in [9] it was taken as $\frac{dp^1\,dp^2\,dp^3}{p_0}$, while in [2] (after their eq. (62)) it was taken as $\frac{dp^1\,dp^2\,dp^3}{p_0}\sqrt{-\det g_{\alpha\beta}}$. With both expressions we obtain the same result for the determination of the fields. But, for accuracy reasons, in the next subsection we will prove that the previous expression is the correct one. It is clear that the authors of [2] forgive that $d\vec{P}$ doesn't denote an integration over all the 4-dimensional space but over an hypersurface of this space, the hypersurface with equation $g_{\alpha\beta}\, p^\alpha p^\beta = m^2c^2$.

A.1 The integral factor $d\vec{P}$.

It is easier if we start with integrals over all $\Re^4$ in the following way: In [8] it was defined
$p^{*\,\alpha} = p^\alpha\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)$; in this way the integration factor can be used $\psi(\mathcal{I})\,d\vec{P^*}$ all over the subspace of $\Re^4$ given by the 4-dimensional cone $p^{*\,\alpha}\,p_{*\,\alpha} \geq m^2c^2$. But, to integrate over this cone, it is convenient to come back to the old variables. Since

$$p^{*\,1} = p^1\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\,,\, p^{*\,2} = p^2\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\,,p^{*\,3} = p^3\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\,, \tag{40}$$

the Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial p^0}{\partial p^1}\left(1+\frac{\mathcal{I}}{m\,c^2}\right) & \frac{\partial p^0}{\partial p^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right) & \frac{\partial p^0}{\partial p^3}\left(1+\frac{\mathcal{I}}{m\,c^2}\right) & \frac{p^0}{m\,c^2} \\ \left(1+\frac{\mathcal{I}}{m\,c^2}\right) & 0 & 0 & \frac{p^1}{m\,c^2} \\ 0 & \left(1+\frac{\mathcal{I}}{m\,c^2}\right) & 0 & \frac{p^2}{m\,c^2} \\ 0 & 0 & \left(1+\frac{\mathcal{I}}{m\,c^2}\right) & \frac{p^3}{m\,c^2} \end{vmatrix} = \tag{41}$$

$$= \frac{1}{m\,c^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^3\left(-p^0 + p^1\frac{\partial p^0}{\partial p^1} + p^2\frac{\partial p^0}{\partial p^2} + p^3\frac{\partial p^0}{\partial p^3}\right).$$

Now the derivatives of $g_{00}\,\left(p^0\right)^2 + 2\,g_{0i}\,p^0p^i + g_{ij}\,p^ip^j = m^2c^2$ with respect to p^k give

$$\left(g_{00}\,p^0 + g_{0i}\,p^i\right)\frac{\partial p^0}{\partial p^k} = -\,g_{0k}\,p^0 - g_{kj}\,p^j \quad \to \quad \frac{\partial p^0}{\partial p^k} = -\frac{p_k}{p_0} \quad \to \quad .$$

$$\to \quad -p^0 + p^1\frac{\partial p^0}{\partial p^1} + p^2\frac{\partial p^0}{\partial p^2} + p^3\frac{\partial p^0}{\partial p^3} = -\frac{p^0\,p_0 + p^k\,p_k}{p_0} = -\frac{m^2c^2}{p_0}.$$

So (41) gives

$$|J| = \left(1+\frac{\mathcal{I}}{m\,c^2}\right)^3\frac{m}{p_0}.$$

So, the integral factor $\psi(\mathcal{I})\,d\vec{P^*}$ becomes

$$\varphi(\mathcal{I})\,\frac{d\,p^1\,d\,p^2\,d\,p^3}{p_0}\,d\,\mathcal{I}.$$

where we have chosen $\psi(\mathcal{I}) = \varphi(\mathcal{I})\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^{-3}\frac{1}{m}$.
So we have found the same expression in [9] for the case of a Minkowsky spacetime; we only have to pay attention to the fact that p_0 isn't equal to p^0, but to $g_{0\alpha}\,p^\alpha$.

A.2 Determination of V^α and $T^{\alpha\beta}$

Let us consider $(29)_1$ for $n = 0$ contracted with U_{α_1} and $(29)_1$ for $n = 1$ contracted with $\frac{U_{\alpha_1}U_{\alpha_2}}{c^2}$. To calculate the integrals we use the change of variables

$$p^1 = \frac{m\,c}{\sqrt{-G_2}}\,\sinh s\,\cos\psi\,,\ p^2 = \frac{m\,c}{\sqrt{-G_3}}\,\sinh s\,\sin\psi\,\cos\phi\,,$$
$$p^3 = \frac{m\,c}{\sin\vartheta\,\sqrt{-G_3}}\,\sinh s\,\sin\psi\,\sin\phi,\ \text{with}\quad s\in[0\,,+\infty[\quad \psi\in[0\,,\pi[\quad \phi\in[0\,,2\,\pi[\,.$$

The Jacobian of the transformation is

$$J = \frac{m^3c^3}{-\sin\vartheta\,G_3\,\sqrt{-G_2}}\,\cosh s\,\sinh^2 s\,\sin\psi\,,\quad \text{and, moreover, we have}\quad p^0 = m\,c\,\cosh s\,,$$
$$U_\alpha p^\alpha = m\,c^2\Gamma\left(\cosh s + \frac{1}{\sqrt{-G_2}}\,\frac{v}{c}\,\sinh s\,\cos\psi\right).$$

The second one of these relations comes out from $p^\alpha p^\beta g_{\alpha\beta} = m^2 c^2$. Eqs $(29)_1$ for $n = 0$ contracted with U_{α_1} and $(29)_1$ for $n = 1$ contracted with $\frac{U_{\alpha_1} U_{\alpha_2}}{c^2}$ give ρc^2 and e; after calculating the integrals in $d\phi$ and in $d\psi$ we can divide the second expression by the first one and obtain (33).

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